Developing a Theoretical Framework for Examining Student Understanding of Fractional Concepts: An Historical Accounting

Susan M. Cooper, Trena L. Wilkerson, Mark Montgomery, Sara Mechell, Kristin Arterbury, Sherrie Moore

Susan M. Cooper, Assistant Professor, University of Wisconsin-Green Bay
Trena L. Wilkerson, Associate Professor, Baylor University
Mark Montgomery, Instructional Specialist, Waco Independent School District, Waco, TX
Sara Mechell, Teacher, Midway Middle School, Hewitt, TX
Kristin Arterbury, High School Mathematics Specialist, Region XII ESC, Waco, TX
Sherrie Moore, Special Education Teacher, Lake Air Middle School, Waco, TX

Abstract

In 2007, a group of mathematics educators and researchers met to examine rational numbers and why children have such an issue with them. An extensive review of the literature on fractional understanding was conducted. The ideas in that literature were then consolidated into a theoretical framework for examining fractions. Once that theoretical framework was in place, it was decided that we would utilize this framework to develop an assessment of student understanding of fractions and to create lessons that would hopefully impact the understanding of the participating students. The intent of this paper is to describe that journey to examine the past research and theory around fractional understanding that lead to a theoretical framework. The paper will also share this framework, as well as provide some information about the instruments and lessons that were created as a result.

Introduction

In the spring of 2007, a group of six mathematics educators came together as part of Baylor University's graduate program to design a course related to mathematics education that would be of value to all six of them. The backgrounds of these six were very different. One was a tenured professor who had conducted research on many different areas of mathematics teaching and learning. Two were middle school teachers; one was still teaching and attending school part time, while the other had left teaching to attend graduate school full time to complete a doctorate in education. One was an elementary school teacher with little formal background in mathematics outside the methods courses required for certification. One was a high school teacher with fifteen years of experience, and one was high school certified but had taught adult remedial education for most of her career. In addition, four of the participants had majored in mathematics during their undergraduate careers, while the other two had majored in elementary education with no specific emphasis on mathematics. At first glance, it would be easy to assume that such a diverse group would struggle to reach consensus on what would be a worthwhile investigation. However, it took only a short time to decide to research rational numbers, specifically fractions. The reason for this was that, at all the different levels with which the six of us worked, we had all experienced issues with our students' understanding of fractions, or lack thereof.

Data from national and international assessments clearly support the existence of the difficulties American students have with fractions we had all observed in our own experiences. For example, the National Assessment of Educational Progress (NAEP), often referred to in the US as the *Nation's Report Card*, has historically shown that students struggle with all but the least complex questions involving fractions. Wearne and Kouba (2000) found in their analysis of the 1996 NAEP assessment that students struggled with problems that were multi-step or non-routine. Kastberg and Norton (2007) furthered this analysis by comparing results from the 1996, 2000 and 2003 NAEPs. Again, students did well on simple questions such as identifying the picture that represents a specified fraction, with 83 percent of tested 4th grade students answering this question correctly in 2003 (89). However, more complex problems such as naming and shading an equivalent fraction remained a struggle for 4th graders, as only 19 percent of students correctly responded correctly (89). An examination of the latest NAEP data shows there are still struggles with these fraction concepts (NCES 2009). In 2009, students improved somewhat in dealing with equivalent fractions, as 55% were able to correctly identify the picture showing that 3/4 and 6/8 are equivalent. However, only 25% could accurately determine which of four fractions was closest to 1/2. Both of these questions were classified as being of low complexity.

An examination of international testing data from the *Trends in International Math and Science Study* (TIMSS) further supports the findings from the NAEP and would seem to indicate that fractions are more of a problem for American children than for children from many other countries. Gonzales et al. (2004) reported results of the 2003 TIMSS study revealing that 4th grade students in the United States had scores that were significantly lower than their counterparts in eleven of the twenty-four participating countries. Only 59% of the questions in the strand related to fractions and number sense were answered correctly on the 1999 TIMSS (Mullis et al. 2000). All this demonstrates further the ongoing trend of the challenges faced by children in the United States regarding fractions.

The final proof of the need for research into student fractional understanding comes from the difficulty teachers themselves have with understanding fractional concepts. This is evidenced most clearly in the research of Liping Ma (1999). Ma demonstrated this difficulty by asking two groups—one of American teachers and one of Chinese teachers—to solve a problem involving division of fractions and to write a word problem that would utilize the given problem as a solution. Within the group of Chinese teachers, all were able to solve the problem, and 90% of them were able to write at least one word problem that was pedagogically correct for division of fractions. In contrast, only 43% of the teachers from the United States were even able to solve the problem presented, and only one was able to write a mathematically accurate word problem.

Why Are Fractions So Difficult?

In considering the difficulties students (and teachers) have with fractions, it is not surprising that much of the research refers to the complex nature of fractional understanding. One of the reasons for this complexity is the relative thinking required to comprehend the meaning of a fraction (Lamon 2008a). When dealing with whole numbers, students are able to apply principles of cardinality to the idea of a number of objects. Cardinality refers to the fact that, when counting a set of objects, the last number said tells the amount of the set (Van de Walle, Karp, and Bay-Williams 2010). A fraction, however, does not represent a specific amount, but rather it represents some portion of an amount. Because of this students must be able to think multiplicatively (in relative terms) rather than additively (in absolute terms). Even this is too simplified a description. The research actually notes many different interpretations of a single fraction such as subdividing an area in equal-sized parts; subdividing a set; as a ratio; and as a way to express division (Lamon 2008a; Charalambous and Pitta-Pantazi 2007; Moss 2005; Witherspoon 2002). Students must be comfortable with all of these interpretations and this way of thinking to have deep fractional understanding, and they must be able to do so without confusing whole number characteristics and fraction characteristics.

Unfortunately, another major issue with student understanding involves the way in which fractions are taught (Lamon 2008a; Chan, Leu, and Chen 2007; Paik and Mix 2003; Cramer, Post, and del Mas 2002; Mack 1995; Bezuk and Cramer 1989). It is a fault of the current system of education in the United States that concepts are often taught using procedures and memorization rather than having students develop their own understanding of fraction concepts. Moss (2005) sums up this issue very clearly with a quote from a student, “Oh fractions! I know there are lots of rules but I can’t remember any of them and I never understood them to start with” (309). Proceeding with this sort of rote memorization can create many different problems. First, this method tends to discount the informal knowledge students already possess with regard to fractions. Students enter school with some understanding of equal sharing and fractions, but this prior knowledge is not always properly accessed and built upon (Mueller, Yankelewitz, and Maher 2010; Empson 1999). Brizuela (2005) further observed this understanding through the examination of children’s use of fractional notation. Rote memorization also does not allow for the connection of mental operations to fraction notations, which is key for real understanding to exist (Saenz-Ludlow 1994), and it can actually be detrimental to a student’s development of numerical reasoning to present an algorithm too early (Kamii 1994).

The variety of concrete models can also be a confounding factor for students as they attempt to make sense of fractions with so many different representations. The research is clear that concrete representations are key for student comprehension of fractions (Van de Walle, Karp, and Bay-Williams 2010; Cramer and Wyberg 2009; Lamon 2008a; Watanabe 1996; Taube 1995; Bezuk and Cramer 1989). The use of manipulatives in teaching fractions is critical, because it accesses Piaget’s levels of development and a student’s need to have physical knowledge of a concept prior to presentation of an algorithm (Bransford, Brown, and Cocking 1999; Kamii and Warrington 1999). There are two categories of concrete models for fractions—

continuous (regions or lines) and discrete (sets of objects) (Van de Walle, Karp, and Bay-Williams 2010)—and the research does not necessarily agree on which model students should be presented with first in their initial introduction to a fraction concept. Some research recommends that students begin with a continuous model, because it is more generalizable to other models (Cramer, Wyberg, and Leavitt 2008; Bezuk and Cramer 1989; Behr, Wachsmuth, and Post 1988). However, the stated issues about some continuous models, such as the difficulty with visual distractors (Cramer, Post, and Behr 1989) and the artistic troubles students have in using continuous models to explain their reasoning (Lamon 2008a), could easily lead the reader to believe the discrete model is the better starting point. There is also research that stresses the need to use both models at all stages of the conceptual development for fractions to increase a student's flexibility in thinking about fractions (Lamon 1996; Watanabe 1996; Taube 1995). With all the conflicting views about how to teach fractions, it is easy to see why students would struggle to learn them.

Our Timeline

Our initial meeting began without the benefit of all the above-mentioned literature. The review of the current research was to be one of the first products. It was therefore necessary to have a point at which to begin investigating, for which we chose a couple of texts that we felt would give us a feel for the seminal works on fractional understanding. One was Susan Lamon's book *Teaching Fractions and Ratios for Understanding* (2008a) and its companion text *More* (2008). This text was chosen in large part because it served the dual purpose of providing research into many of the areas that cause issues with learning fractions as well as supplementing that research with both examples of student work and with strong problem examples that could be used to deepen our comprehension of issues with fractions. The other text was *Making Sense of Fractions, Ratios, and Proportions* (NCTM 2002), the 64th yearbook from the National Council of Teachers of Mathematics (NCTM) that focused specifically on recent research related to issues with rational numbers and offered us a wider view of the existing research.

While reading through these two texts, each of the participants also conducted an expanded investigation on a topic of personal interest related to fractions that would then be shared in detail with the group. For example, one participant examined the work of Joan Moss (2002) suggesting that the teaching of rational numbers should be reordered to begin with concepts of percents rather than with fractions. Another participant considered the physical models that are used to represent fractions, such as the line model, continuous model, and discrete model, and analyzed the research regarding the ramifications of each of these choices. A third member of the group extended Brizuela's (2005) work and considered fraction notation and its significance in student understanding.

Along with the extensive examination of theory and research on fractions we conducted throughout the spring, we also decided it was critical to consider how fractional concepts appeared in our own state's standards. Texas utilizes the Texas Essential Knowledge and Skills

(TEKS) (Texas Education Agency 2007) as its statewide curriculum, and it is these skills students are tested on at the end of every school year. We also felt it was important to see how this curriculum connected to the newest national standards via the *Curriculum Focal Points* (2006) that were released by NCTM. What we discovered that was central to the eventual research we conducted was a disparity between the state's curriculum at the time and the recommendations made by NCTM (see Figure 1). While NCTM makes no mention of fractions or fractional concepts until third grade, the TEKS had students begin working on fractional language, unit fractions, fair sharing, and parts of a whole as early as kindergarten.

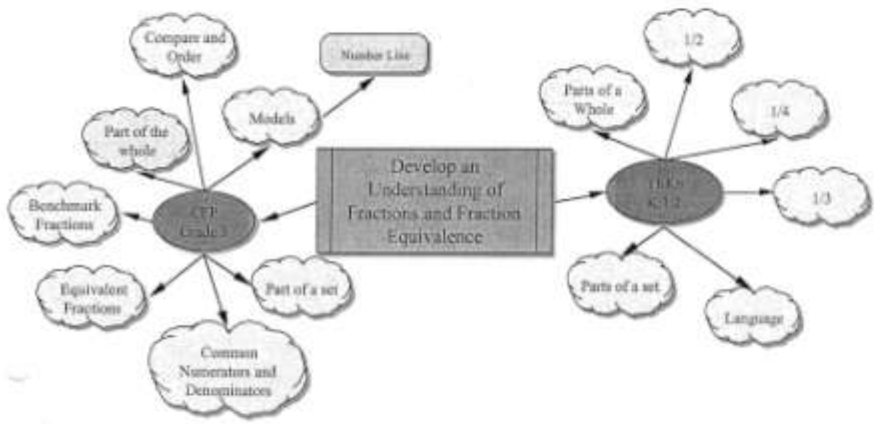

Figure 1. Curriculum Focal Points (CFP) compared to the Texas Essential Knowledge and Skills (TEKS)

This was only one of many contradictions in the research that piqued our curiosity about how students actually learn fractions most effectively. It was partly this curiosity that provided fertile soil for the growth of a formalized research project once the seed was planted. Baylor University in Waco, Texas, USA, has a tradition of conducting research in partnership with the local school district. For example, as part of its teacher education program, all of the pre-service teachers must conduct at least one action research project within their various field placements as a requirement of graduation. In the past, more formalized research had also been conducted in cooperation with one of the local schools on improving the understanding of geometric concepts in the early grades. When it was discovered that a cohort was studying student struggles with fractions, it was therefore natural for the principal of this school to ask if there was any intent to turn this into a research study similar to the geometry study that had been conducted at the school previously. We agreed to this request as a way to further our own understanding of the issues with teaching fractions in the early grades. The decision was made to focus on early learning because the one issue that all the research seemed to agree on was that a foundation in the understanding of what a fraction is and how different fractions relate to each other was critical for success in any of the other rational number concepts.

Theoretical Framework

As with any research project, the first goal was to synthesize all of our learning thus far into a theoretical framework for a research study on fractions. Although there were many different areas of study to consider within the data that was being collected, it was decided that the most important aspect of the study was to first examine student understanding. Over the range of grades from kindergarten through third, this understanding was segmented into four distinct areas that appeared critical to later fraction success. These were fair sharing, part-whole partitioning, unitizing, and equivalence. These four concepts became the framework around which we developed our research.

Fair Sharing

At the absolute foundation of fractional understanding is the ability of a student to share some object or objects fairly into a preset number of divisions. A key understanding about fair sharing is that it accesses a student's experiences outside of school (Lamon 2008a; Empson 1999). Students have many opportunities in life to fairly share things with friends and siblings, and many have even experienced how to deal with fair sharing when splitting something that did not come out evenly, such as two cookies among three children. Lamon notes that the difficulty in fair sharing is for students to recognize that equal refers to the amount given, not necessarily the dimension or number of pieces.

Part-Whole Partitioning

Part-whole relationships are defined as those in which a student "designates a number of equal parts of a unit out of the total number of equal parts into which the unit is divided" (Lamon 2008a 125). Once students are able to competently apply fair sharing skills to both discrete and continuous objects, quantifying those shares becomes critical. This is accomplished through part-whole understanding and the structure of a fraction. Cramer and Wyberg (2009) specify that understanding of the part-whole relationship "relies on the comparison between 'shaded part' and the whole unit" (229). While this may seem a simple concept, the thinking required is quite complex. As Norton and Wilkins (2009) note, "This scheme relies upon operations of identifying (unitizing) a whole, partitioning the whole into equal pieces, and disembedding some number of pieces from the partitioned whole" (2). As noted by Charalambous and Pitta-Pantazi (2007) the concept of part-whole partitioning takes up the bulk of the curriculum in younger grades, because it is critical to understanding other rational number concepts such as ratios, quotients, and measure. Behr, Lesh, Post, and Silver (1983) summarize this in their research for the *Rational Number Project* by stating, "It seems plausible that the part-whole subconstruct, based both on continuous and discrete quantities, represents a fundamental construct for rational-number concept development" (10).

Equivalence

Equivalence with fractions refers to the fact that many different fractions can be used to name the same quantity, depending on how the quantity is subdivided (Lamon 2008a). This need to be able to subdivide a quantity creates its own unique issues. Cramer, Post, and Behr (1989) discuss the issue of perceptual distractors in their research, which are created by these very subdivisions. When considering an area divided into six regions with four of those regions shaded, students are often unable to recognize that this picture can represent both the fractions four-sixths and two-thirds.

Further Specifications

Once these larger concepts were chosen, it was still necessary to further specify what we would be looking for as we designed the study and developed coding structure to analyze the data that would be collected. It was decided that, first and foremost, the model being used to teach a concept would be critical in analyzing student success. As was stated earlier, providing students with concrete representations is one of the components that is key to a student being successful in working with fractions and having a strong understanding of fraction concepts (Van de Walle, Karp, and Bay-Williams 2010; Cramer and Wyberg 2009; Lamon 2008a; Watanabe 1996; Taube 1995; Bezuk and Cramer 1989). The two categories of models used for fractions are continuous, meaning they can be subdivided into smaller pieces, and discrete, which consists of a group of undividable objects. Van de Walle, Karp, and Bay-Williams (2010) further classify continuous models as area models and length models, although we did not utilize this subdivision for the purpose of coding. This was in large part due to the fact that the emphasis on ordering fractions did not occur until later grades. We also included a category for other models in order to allow for flexibility in coding what students did in the event their actions did not fit directly into the use of a continuous or a discrete model.

The observed actions within each of these models was then further subdivided to analyze the types of representations used and the types of understanding that manifested during the research. We specifically considered whether students used language, pictures, symbols, or actions to represent their fractional understanding. With regard to understanding, it was decided that we would need to consider both understanding and misunderstanding specific to the issue being observed, as well as to analyze a student’s responses for deeper misconceptions about fractions. All these pieces are summarized in Figure 2, and it was this that made up the developed theoretical framework for our research on student fractional understanding.

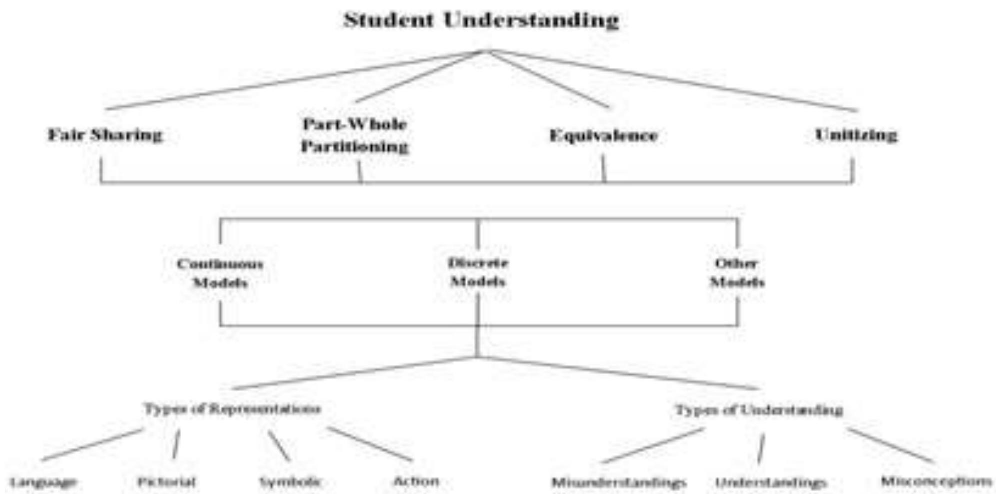

Figure 2. Theoretical Framework for Student Fractional Understanding

Project Development

The theoretical framework was only the beginning of what has become a much larger longitudinal study of fractional understanding. Utilizing this developed framework as the foundation, the team met in the summer to design the research study requested by the principal. Because we were dealing with early fractional understanding, it was decided that the research would begin by working with kindergarten and third grade students—the bookends of this understanding as defined by our theoretical framework. This process consisted first of the development of an interview instrument to use both pre- and post-treatment to assess the individual student's understanding of fractions. The questions for these assessments were modified from a variety of sources, including the Texas Assessment of Knowledge and Skills (TAKS), the California Achievement Test (CAT), the Iowa Test of Basic Skills (ITBS), and the problems available for consideration in Lamon's book (2008a). The questions chosen aligned both with our theoretical framework and with the expected grade level skills as outlined previously in Figure 1. The protocols were designed to be given orally and individually and were scored using four point rubrics that reflected the research about what constituted complete and partial fractional understanding.

Once the instruments were developed, lessons were written that again specifically addressed the components of the theoretical framework. Because of school scheduling constraints, these lessons were only taught once a week for approximately six weeks in small group settings. Each of the lessons strived to exemplify the tenets of best practice by having students work in contextual settings with both discrete and continuous models for fraction learning. The lessons were taught by junior-level and senior-level pre-service teachers who were in the school as part of the University's teaching certification program. These lessons were observed by both professors and graduate students from Baylor University with the intent of finding evidence of student fractional understanding as laid out in the framework.

Although student understanding was the initial focus of the research, other data was collected on the teacher candidates as well, since their own knowledge of fractions and rational numbers is key to the success of the students they will eventually teach. The teacher candidates wrote reflective journals regarding their assessment of the day's lesson and how well students understood what was being taught. They were also asked to take a pre- and post-assessment of their own fractional understanding, and they completed the *Mathematics Teacher Efficacy Beliefs Instrument (MTEBI)*. This data will eventually be analyzed for evidence of changes in efficacy for teaching fractions, as well as for changes in teacher content knowledge (TCK) and pedagogical content knowledge (PCK) for mathematics.

Currently, this study is in its fifth iteration. Instruments and lessons have been developed for first and second grade students as well. Modifications have been made to lessons to improve them for student learning. The data is also being analyzed from a longitudinal standpoint, as some of the participants were in the study for all four grade levels. Informal results are positive, both anecdotally from the classroom teachers of participating students and from the data analyzed thus far.

Despite all that has been accomplished with this study, it is important to note that this was very much a focused study based strictly on student fractional understanding. While a more systemic investigation of student mathematical understanding may have been more meaningful for explaining student difficulties, the point of this study was to examine within the existing framework of a classroom how to improve student understanding of fractions. The researchers recognize that a student's mathematical understanding of more basic concepts is going to have an impact on how well he or she does with fractions. However, the purpose of this study was to consider ways to improve fractional understanding rather than to create better overall mathematics instruction.

Conclusion

Because of the traditional methods of rote instruction that have dominated American mathematics education in the past, there are many areas in which both teachers and students struggle to reach a deeper conceptual understanding. Fractions seem to be one of those predominant areas for many. Through this study, the six researchers and by extension the participating teachers and pre-service teachers were able to gain a stronger understanding of the nuances within the learning of fractions. It is this sort of understanding that is required in order for change to be effected and instruction on fractions to be improved.

Of potentially more significance is the development of the actual research itself. Rather than there being a top-down approach to change, practitioners were given the power to research a problem of concern to them and work toward a potential solution. These practitioners were then able to bring their research to others in the classroom to determine its effectiveness. While some may resist research that does not follow a traditional research design, this sort of model is one that should be considered for more areas and be put to wider use than just through the facilitation of a university. In the spirit of action research or Japanese lesson study, this historical accounting can perhaps be used as a model for allowing practitioners to develop solutions to the problems they face with content instruction as an alternative to current models.

References

Behr, M.J., I. Wachsmuth, and T. Post. 1988. Rational number learning aids: Transfer from continuous models to discrete models. *Focus on Learning Problems in Mathematics* 10: 1-17.

Bezuk, N. and K. Cramer. 1989. Teaching about fractions: What, when, and how? In *National Council of Teachers of Mathematics 1989 Yearbook: New Directions for Elementary School Mathematics,* edited by P. Trafton. Reston, VA: National Council of Teachers of Mathematics.

Bransford, J.D., A.L. Brown, and R.R. Cocking (Editors). 1999. *How people learn: Brain, mind, experience, and school.* Washington DC: National Academy Press.

Brizuela, B.M. 2005. Young children's notations for fractions. *Educational Studies in Mathematics* 62: 281-305.

Chan W.H., Y.C. Leu, and C.M. Chen. 2007. Exploring group-wise conceptual deficiencies of fractions for fifth and sixth graders in Taiwan. *The Journal of Experimental Education* 76: 26-57.

Charalambous, C. and D. Pitta-Pantazi. 2007. Drawing on a theoretical model to study students' understanding of fractions. *Educational Studies in Mathematics* 64: 293-316.

Cramer, K.A., T.R. Post, and M.J. Behr. 1989. Cognitive restructuring ability, teacher guidance, and perceptual distracter tasks: An aptitude-treatment interaction study. *Journal for Research in Mathematics Education* 20: 103-110.

Cramer, K.A., T.R. Post, and R.C. delMas. 2002. Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the Rational Number Project curriculum. *Journal for Research in Mathematics Education* 33: 111-144.

Cramer, K. and T. Wyberg. 2009. Efficacy of different concrete models for teaching the part-whole construct for fractions. *Mathematical Thinking and Learning* 11: 226-257.

Cramer, K. t. Wyberg, and S. Leavitt. 2008. The role of representations in fraction addition and subtraction. *Mathematics Teaching in the Middle School* 13: 490-496.

Empson, S.B. 1999. Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction* 17: 283-342.

Gonzales, P., J.C. Guzman, L. Jocelyn, D. Kastberg, T. Williams. 2004. *Highlights from the Trends in International Mathematics and Science Study (TIMSS) 2003.* Washington DC: Institute of Education Sciences.

Kamii, c. and M.A, Warrington. 1999. Teaching fractions: Fostering children's own reasoning. *Developing Mathematical Reasoning in Grades K-12,* (82-92), edited by V.L. Stiff and R. F. Curcio. Reston, VA: National Council of Teachers of Mathematics.

Lamon, S.J. 1996. The development of unitizing: Its role in children's portioning strategies. *Journal for Research in Mathematics Education* 27: 170-193.

Lamon, S.J. 2008. *More: In-depth discussion of the reasoning activities in "Teaching fractions and ratios for understanding".* New York: Taylor & Francis Group.

Lamon, S.J. 2008a. *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers (2nd ed.).* New York: Taylor & Francis Group.

Litwiller, B. and G. Bright (Editors). 2002. *Making sense of fractions, ratios, and proportions: 2002 yearbook.* Reston, VA: National Council of Teachers of Mathematics.

Ma, L. 1999. *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* London: Lawrence Erlbaum Associates, Publishers.

Mack, N.K. 1995. Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education* 26: 422-441.

Moss, J. 2005. Pipes, tubes, and beakers: New approaches to teaching the rational-number system. In *How People Learn: Brain, Mind, Experience and School,* (309-349), edited by J.D.. Bransford, A.L. Brown, and R.R. Cocking. Washington DC: National Academy Press.

Moss, J. 2002. Percents and proportion at the center: Altering the teaching sequence for rational number. In *Making Sense of Fractions, Ratios, and Proportions* (109-120), edited by B. Littweiller and G. Bright. Reston, VA: National Council of Teachers of Mathematics.

Mueller, M., D. Yankelewitz, and C. Maher. 2010. Rules without reason: Allowing studnets to rethink previous conceptions. *Montana Mathematics Enthusiast* 7: 307-320.

Mullis, I.V.S., M.O. Martin, E.J. Gonzales, K.D. Gregory, R.A. Garden, K.M. O'Connor, et al. 2000. *TIMSS 1999 international mathematics report: Findings from IEA's repeat of the Third International Mathematics and Science Study at the eighth grade.* Chestnut Hill, MA: Boston College.

National Center for Education Statistics. 2009. *NAEP Questions Tool.* Washington, DC: Institute of Education Sciences. Accessed February 24, 2012. http://nces.ed.gov/nationsreportcard/itmrlsx/.

National Council of Teachers of Mathematics (NCTM). 2006. *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence.* Reston, VA: NCTM.

Norton, A. and J. Wilkins. 2009. A comparison of the part-whole and partitive reasoning with unit and non-unit proper fractions. In *Proceedings of the Thirtieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* edited by D.Y. White. Atlanta: Georgia State University.

Paik, J.H. and K.S. Mix. 2003. U.S. and Korena children's comprehension of fraction names: A reexamination of cross-national differences. *Child Development* 74: 144-154.

Saenz-Ludlow, A. 1994. Michael's fraction schemes. *Journal for Research in Mathematics Education* 25: 50-85.

Taube, S.R. 1995. Reconstructing the whole: A gauge of fraction understanding. *Paper presented at the 17th annual meeting of the North American Chapter for the Psychology of Mathematics Education.* Columbus, OH.

Texas Education Agency (TEA). 2007. *Texas essential knowledge and skills by chapter: Subchapter A. Elementary.* Retrieved December 8, 2007 from http://www.tea.state.tx.us/rules/tac/chapter111/ch111a.html.

Van de Walle, J.A., K.S. Karp, and J.M. Bay-Williams. 2010. *Elementary and middle school mathematics: Teaching developmentally.* Boston: Allyn & Bacon.

Watanabe, T. 1996. Ben's understanding of one-half. *Teaching Children Mathematics* 2: 460-464.

Witherspoon, M.L. 2002. Fractions: In search of meaning. In *Putting Research into Practice in the Elementary Grades: Readings from Journals of the National Council of Teachers of Mathematics,* edited by D.L. Chambers. Reston, VA: National Council of Teachers of Mathematics.

Published by the Forum on Public Policy